Received: *May 29, 1998*, Accepted: *June 9, 1998*

Quark potentials in the Higgs phase of large N supersymmetric Yang-Mills theories

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Abstract: We compute, in the large N limit, the quark potential for $\mathcal{N}=4$ supersymmetric $SU(N)$ Yang-Mills theory broken to $SU(N_1)\times SU(N_2)$. At short distances the quarks see only the unbroken gauge symmetry and have an attractive potential that falls off as $1/L$. At longer distances the interquark interaction is sensitive to the symmetry breaking, and other QCD states appear. These states correspond to combinations of the quark-antiquark pair with some number of W-particles. If there is one or more W-particles then this state is unstable because of the Coulomb interaction between the W-particles and between the W's and the quarks. As L is decreased the W-particles delocalize and these Coulomb branches merge onto a branch with a linear potential. The quarks on this branch see the unbroken gauge group, but the flux tube is unstable to the production of W-particles.

Keywords: Nonperturbative Effects, Spontaneous Symmetry Breaking, 1/N Expansion, Brane Dynamics in Gauge Theories.

JHEP06(1998)005

Contents

1. Introduction

The conjectured link [1] between large N $\mathcal{N} = 4$ supersymmetric Yang-Mills and ten dimensional supergravity on $AdS_5 \times S_5$ has led to a number of interesting predictions. In particular one has been able to compute the large N behavior for the attractive potential between heavy quarks and antiquarks, both at zero temperature [2, 3], and finite temperature [4, 5, 6], and for a quark and a monopole [7], collections of dyons, monopoles and quarks [8], and flux tubes [9]. In these calculations, one looks for static string configurations that correspond to a separation L and potential energy E between the heavy particles.

For a quark-antiquark pair at zero temperature there is a unique string configuration for a given separation L [2, 3]. At finite temperature, one finds that there are some values of L where two solutions exist [5, 6]. The branch with the lower energy is the Coulomb branch, where at short distances, the quark and anti-quark experience an attractive Coulombic force. The other branch is a "confining" branch, in that the force is linear in L, at least for small L. The physical meaning of this branch is unclear.

There are other systems where more than one branch exists. As we will see below, an $SU(N)$ gauge theory broken to $\prod_i SU(N_i)$ with gN_i large will have many branches. In fact as L diverges, the number of branches diverges. In this paper we will study the breaking of $SU(N)$ to $SU(N/2) \times SU(N/2)$. In this situation some of the string configurations can be solved by quadratures, or exactly in various limits, and the qualitative features can be generalized to other breakings.

The physics of these extra branches can be deduced by looking at the energy density along the static string configuration, and also by taking the quark separation to infinity. In particular we find static string configurations that correspond to states consisting of a quark, an antiquark and n massive W particles, with the latter transforming in the $(N/2, \overline{N}/2)$. The energy density along the string concentrates appropriately: there are two (infinitely) massive quarks at the ends and n equal and evenly spaced mass concentrations in between. In the limit of infinite separation the total energy of the system becomes the sum of the BPS quark masses plus the BPS masses of n W-particles, while the energy density reduces to that of the quarks and n evenly spaced δ-functions for the W-particles. The static string configurations in this limit also decompose into a concatenation of such configurations for a quark, an antiquark and n intermediate W's. As the quark separation for the n W-particle configuration is decreased the branch becomes unstable to small perturbations because of the attractive Coulomb potential between the W particles and the quarks. As L is decreased further the W-particles delocalize and the unstable branches merge onto a branch with a potential that is linear for large enough L.

Turning this around, for large enough quark separation there is an unstable flux tube configuration that has a linear potential between the quarks. This is an unbroken gauge configuration in that the quark and anti-quark are both in the fundamental of $SU(N)$. The flux tube can then destabilize by popping out W-particles that are then subject to the Coulomb instability. Eventually, a stable configuration is reached where we have a quark and an anti-quark in $(N/2, 1)$ or $(1, N/2)$ representations of the broken group. Finally, as L is decreased even further, these stable branches merge onto the unbroken gauge branch. For L below this critical value this is the only string solution.

In section 2 we discuss some general features for static string configurations and discuss how the study of such configurations reduces to the study of particle motion in a potential, V. We use Maupertuis' principle to argue that the force between the quarks is always attractive and has a simple relation to the classical particle's energy. We then specialize to geodesics that correspond to an $SU(N)$ gauge theory broken to $SU(N/2) \times SU(N/2)$, for which the potential, V, is bipolar and $\mathbb{Z}_2$ symmetric. In section 3 we consider string trajectories that lie on the symmetry axis. We show that there is a branch of trajectories that runs up a saddle and which is Coulombic for small separations and confining for large. We also show that there are infinitely many branches that merge onto this confining branch. All but the lowest energy branches are unstable to small perturbations and eventually become Coulombic as the separation is increased. In section 4 we consider trajectories that are off the symmetry axis. In this case, the quark and anti-quark have unequal masses. The branches split into groups of three, except for the lowest energy branch which is isolated. The higher energy branches have a minimum quark separation. Pushing the quarks inside of this separation leads to a finite release of energy. Section 5 contains some final comments.

2. Metrics, geodesics and the quark potential

2.1. Generalities

The general form of the metric for a configuration of parallel, static D3-branes in ten dimensions is

$$ds^2 \;=\; H^{-1/2}\, dz^\mu dz^\nu \eta_{\mu\nu} \;+\; H^{1/2}\, dy^m dy^m\,, \tag{2.1}$$

where $\mu = 0, \ldots, 3$ and $m = 4, \ldots, 9$, and H is a function of the y^m alone.

Our purpose is to use the ideas first mooted in [1], and further developed in [2, 3, 4, 5, 6, 7, 8, 9], to study the quark-antiquark potential. This means that we consider static string configurations that begin and end on a single brane at a very large (and ultimately infinite) distance, $\vec{y} = \vec{y}_{max}$, from the rest of the branes in the configuration defined by H. Finding such static string configurations is equivalent to finding geodesics in the spatial 9-metric:

$$ds^2 \;=\; H(y)^{-1}\, dz^\alpha dz^\alpha \;+\; dy^m dy^m\,, \tag{2.2}$$

where z^α, $\alpha = 1, 2, 3$, runs over the spatial coordinates of the D3-brane. Because of the symmetry in the z^α, we can choose coordinates so that the geodesic runs in the $z^1 = z$ direction, with $z^2 = z^3 = 0$. Let λ be an affine parameter for geodesics in the metric (2.2), then the translational symmetry implies that $\frac{dz}{d\lambda} = const\; H(y)$. We will normalize the affine parameter by taking the constant equal to one, and hence

$$\frac{dz}{d\lambda} \;=\; H(y). \tag{2.3}$$

Using this conservation law in the geodesic equations for $y^m(\lambda)$ yields

$$\frac{d^2 y^m}{d\lambda^2} \;+\; \frac{1}{2}\frac{\partial H}{\partial y^m} \;=\; 0\,. \tag{2.4}$$

We are thus studying the motion of a particle in the (repulsive) potential $V \equiv \frac{1}{2}H$. We will denote the classical energy of this particle by

$$\mu \;=\; \frac{1}{2}\Big(\frac{dy^m}{d\lambda}\Big)^2 \;+\; \frac{1}{2}H(y)\,. \tag{2.5}$$

The energy of the quark-antiquark pair is given by

$$E \;=\; \frac{1}{2\pi\alpha'}\int\sqrt{\Big(\frac{dy^m}{d\lambda}\Big)^2 \;+\; H(y)}\;\; d\lambda \;=\; \frac{1}{2\pi\alpha'}\,\sqrt{2\mu}\;\lambda\,, \tag{2.6}$$

and the three-dimensional separation, L, of the quark-antiquark pair is given by

$$L \;=\; \int H(y(\lambda))\; d\lambda\,. \tag{2.7}$$

The energy density along the string can be readily deduced from (2.6) and (2.7), with the density ρ given by

$$\rho \;=\; \frac{1}{2\pi\alpha'}\,\frac{\sqrt{2\mu}}{H(y)}\,. \tag{2.8}$$

We ultimately wish to study E as a function of L, and seek out Coulombic and confining behavior. One can gain some very useful insight into $E(L)$ by applying some of the variants of variational principles associated with particle motion, most particularly, Maupertuis' principle [10]. Consider a general variation of the action $S(\frac{d\vec{q}}{dt},\vec{q})$ about a solution of the Euler-Lagrange equations, but for which we let the endpoints and the time for the path vary:

$$\delta S \;=\; (\vec{p}_f\cdot\delta\vec{q}_f - H_f\delta t_f) \;-\; (\vec{p}_i\cdot\delta\vec{q}_i - H_i\delta t_i)\,, \tag{2.9}$$

where the subscripts i and f denote initial and final quantities, while $\vec{p}$ is the particle momentum, and H is the Hamiltonian. We will consider variations for which $\vec{p}\cdot\delta\vec{q}=0$, and in partcular variations in which the endpoints do not move. Since the system has time translation invariance with a conserved energy, μ, we have $\delta S=-\mu\delta t$. For such a system, it immediately follows that the particle path is obtained from the variational principle $\delta S_0=0$, where S_0 is the *abbreviated action*:

$$S_0 \;=\; S+\mu(t_f-t_i) \;=\; \int \vec{p}\cdot d\vec{q}\,. \tag{2.10}$$

In doing this variation one considers all paths with a fixed energy, μ, and one must express the integrand of S_0 in terms of $\mu,\vec{q}$ and $d\vec{q}$ only. For example, for a particle moving in a potential V, one has

$$S_0 \;=\; \int\sqrt{2(\mu-V)d\vec{q}\cdot d\vec{q}}\,. \tag{2.11}$$

Consider now, the variation of the path with respect to the energy, μ. The time taken along the path will change in response, and so we have from (2.10):

$$\delta S_0 \;=\; \delta S \;+\; (t_f-t_i)\delta\mu \;+\; \mu\delta t \;=\; (t_f-t_i)\delta\mu\,, \tag{2.12}$$

and so $\frac{\partial S_0}{\partial\mu} \;=\; (t_f-t_i)$.

Returning to the original problem, we want to consider how the quark energy, E and separation, L, change as we vary the geodesic, and in particular, vary the classical particle energy, μ. From (2.6), (2.7) and (2.12), one has

$$\begin{aligned}\delta E &= \frac{1}{\pi\alpha'\sqrt{2\mu}}\,(\,\mu\,\delta\lambda \;+\; \tfrac{1}{2}\lambda\,\delta\mu\,)\,,\\ \delta L &= 2\delta\int V \;=\; \delta(S_0-2S) \;=\; \lambda\delta\mu \;+\; 2\mu\delta\lambda\,.\end{aligned} \tag{2.13}$$

From this one immediately sees that

$$\delta E \;=\; \frac{1}{2\pi\alpha'\sqrt{2\mu}}\,\delta L\,, \qquad \text{or} \qquad \frac{dE}{dL} \;=\; \frac{1}{2\pi\alpha'\sqrt{2\mu}}\,. \tag{2.14}$$

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Thus the force between a quark and anti-quark is determined by the energy of the classical particle.

There are several important consequences of this. First, the force is always attractive: something that is obvious from QCD, but not at all obvious from the string perspective. In particular, if one considers E and L as a function of some other parameter, as one does in the thermal case [5, 6], one sees that E and L must reach a minimum or maximum at the same point. Hence if one of these functions "turns over" then so must the other. Such simultaneous turning points of E and L are crucial to the cusp-like behavior $E(L)$ at bound state transitions. This will be discussed in more detail in section 4.

On a more fundamental level, to compute the quark potential one considers a family of geodesics going out to the brane at infinity. A confining family is thus characterized by a family of such geodesics for which the classical particle energy is constant. One can directly verify this in a special case: the situation where the family consists of a particle that is approaching a local maximum, or saddle point from below, and turning around and reversing its course just before the maximum, or saddle. Suppose the potential at the maximum or saddle is μ_0, and consider a particle coming in with energy $\mu_0 - \epsilon$. For the particle to approach the maximum, the time taken diverges as $t \sim -log(\epsilon)$, and so the inter-quark energy E, and length L, also both diverge as $-log(\epsilon)$, which means that $E \sim L$.

It is also interesting to consider the behavior of E and L for particles that approach, or are released from a point where the potential is diverging as $V(q) \sim q^{-r}$. One can easily see that $L \sim \mu^{\frac{r-2}{2r}}$ and $E \sim \mu^{-\frac{1}{r}}$ as one approaches the singularity.[1] One thus finds $E \sim L^{-\frac{2}{r-2}}$, which is consistent with (2.14), and shows Coulombic behavior only for $r = 4$. We can also find the energy density along z, where we see that the density is $\rho \sim \mu^{-1/2}$ for the part of the string near the peak and $\rho \sim \mu^{1/2}$ for the part that is away from the peak. Hence, very energetic particles correspond to string configurations where the energy density is highly clumped.

2.2. Multi-centered D3-branes

In [1] a single group of N D3-branes was considered, and the function H was given by

$$H \ = \ 1 \ + \ \frac{4\pi g N \alpha'^2}{r^4} \,, \qquad r \ \equiv \ \sqrt{y^m y^m} \,. \tag{2.15}$$

In the large N limit, one then has the anti-de Sitter metric

$$ds^2 \ = \ \frac{U^2}{R^2} \, dz^\mu dz^\nu \eta_{\mu\nu} \ + \ R^2 \frac{dU^2}{U^2} \ + \ R^2 d\Omega_5^2 \,, \tag{2.16}$$

where $U = r/\alpha'$, and $R = (4\pi g N)^{1/4}$. It was also pointed out in [1] that one could equally well consider separated groups of D3-branes, and that for large N and large U these would also limit to anti-de Sitter space.

[1] One must, of course, subtract off the quark mass to get this fall off in the energy.

The multi-centered D3-brane solution is then given by taking [11, 12]

$$H(y) \;=\; 1 \;+\; \sum_a \frac{4\pi g N_a \alpha'^2}{|\vec{y} \;-\; \vec{y}_\alpha|^4}\,, \tag{2.17}$$

where N_a is the number of branes located at the point $\vec{y}_a$. As in [1], we consider the limit of the metric in which $\alpha' \to 0$ with r/α' finite, and in which all of the N_a are large (but N_a/N_b is finite). This means that we may drop the 1 in (2.17). The IIB superstring in this background should then describe a Higgs phase of an $N = 4$ supersymmetric $SU(N)$ gauge theory, with $N = \sum_a N_a$, where the gauge group has been broken to $\prod_a SU(N_a)$. The Higgs vevs are represented by the separations of the groups of D3-branes. In particular, at large $r = \sqrt{y^m y^m}$, the function H in (2.17) takes the form (2.15) with $N = \sum_a N_a$, and thus at large distances, and for large N_a, the multi-centered solution limits to the anti-de Sitter solution corresponding to $SU(N)$.

Our major focus will be the breaking of $SU(N)$ to $SU(M) \times SU(M)$, $M = N/2$, and upon geodesics in the corresponding double-centered metric. By rotating the coordinates, we can take the potential $V = \frac{1}{2}H$ to be

$$V(x,y) \;=\; \frac{\alpha'^2}{4}\left[\frac{R^4}{((x-\alpha'\phi)^2+y^2)^2} \;+\; \frac{R^4}{((x+\alpha'\phi)^2+y^2)^2}\right], \tag{2.18}$$

where we have set $y^1 = x, y^2 = y$. To make contact with the conventions of other authors [1, 2, 3, 4, 5, 6, 7, 8, 9], in whose work α' has been scaled away and in which the variables x and y have dimensions of mass, one makes the rescaling $x \to \alpha' x$, $y \to \alpha' y$ and $\lambda \to \alpha'^2 \lambda$.

We will analyze the geodesic structure extensively in the next section, but here we note some general features. First, for large $r = \sqrt{x^2+y^2}$, one has $V = \alpha' 2R^4/(2r^4)$, and provided the geodesic stays out at large values of r, $(r >> \alpha'\phi)$ the properties of such a geodesic will not differ significantly from those discussed in [2, 3]. In the language of gauge theory this means that the symmetry breaking mass scale of $SU(N)$ to $SU(M) \times SU(M)$, which is proportional to ϕ, is much less than the energy scale of the quark-antiquark interaction $(1/L)$, and so the interaction behaves as it would in the unbroken $SU(N)$ phase.

At the other extreme, there are two geodesics that represent single quarks: That is, the geodesics start on the single D3-brane and terminate on one or other of the groups of M D3-branes. They traverse no distance in the z-direction (the constant in the conserved quantity is zero) and run with infinite classical particle speed. Such geodesics represent quarks in the $(M,1)$ or $(1,M)$ of $SU(M) \times SU(M)$ and the quark mass is simply the distance in $\vec{y}$-space from $\vec{y}_{max}$ to the location of the relevant group of branes.

There is a third important geodesic: it also does not move in the z-direction, but runs directly from one group of M D3-branes to the other. The corresponding classical particle runs from one peak to the other through the interconnecting saddle. This can be thought of as representing the massive W-particle coming from the broken gauge symmetry, and lying in the $(M, \overline{M})$ or $(\overline{M}, M)$ representation depending upon the orientation of the geodesic. The mass of this W is given by (2.6), but without the potential term (the constant of integration for the z-motion is zero) and is thus given by $m_W = \phi/\pi$.

Returning to the geodesics that correspond to a quark-antiquark pair, we will see in the next section, that as we separate the pair, *i.e.* increase L, the geodesic drops closer to the double-centered core. Initially the pair is in an unbroken phase, but at a critical separation there is a phase transition and one can begin to distinguish the $(1, M)$ and the $(M, 1)$. Associated with this transition, one also finds geodesics that correspond to states with the quark, anti-quark and several $(M, \overline{M})$ or $(\overline{M}, M)$ W-particles.

3. Analysis of trajectories

Using the potential derived in the preceding section, we can now analyze the quark-antiquark potential. We first consider trajectories that come in at, or infinitessimally close to an angle of 90 degrees with respect to the $SU(M) \times SU(M)$ scalar axis. The advantage of doing it this way is that the $\mathbb{Z}_2$ symmetry is not broken by the $U(1)$ scalar expectation value. Moreover, the lowest trajectory can be solved by quadratures.

3.1. The geodesic on the symmetry axis

Consider the trajectory that is aimed exactly at the saddle of the potential. The trajectory then stays at 90 degrees since there is no lateral force on the particle. From the potential in (2.18), we see that we have an effective potential in y only, while the x coordinate stays fixed. Using (2.7) and (2.6), the length L on the $D3$ brane between the quarks is

$$L \;=\; 2\int d\lambda\; V(0, y(\lambda))\,, \tag{3.1}$$

while the quark energy is given by the integral

$$E \;=\; \frac{1}{2\pi}\int d\lambda \sqrt{(\partial_\lambda y)^2 + 2V(0, y(\lambda))}\,, \tag{3.2}$$

where the term inside the square root is twice the conserved energy for the particles motion. (We have passed to the conventions of [1] by recaling $x \to \alpha' x$, $y \to \alpha' y$ and $\lambda \to \alpha'^2 \lambda$ as described earlier.)

Using (2.3) and conservation of the particle energy, we see that as a function of z, y satisfies

$$4V^2(0, y)(\partial_z y)^2 = C - 2V(0, y)\,, \tag{3.3}$$

where C is an integration constant. Hence, for a particle tractory that starts at $y = \infty$ goes down to $y = y_0$ and then turns around and goes back to $y = y_0$, we have

$$L = \int_{y_0}^{\infty} \frac{V(0, y)}{\sqrt{2V(0, y_0) - 2V(0, y)}} dy = \tag{3.4}$$

$$= 2R^2(\phi^2 + y_0^2)\int_{y_0}^{\infty} \frac{dy}{(y^2+\phi^2)\sqrt{(y^2+\phi^2)^2 - ({y_0}^2+\phi^2)^2}}\,.$$

The above integral is a combination of elliptic integrals of the first and third kind. In terms of generalized hypergeometric functions the length between the quarks is

$$\begin{aligned} L = & \frac{(2\pi)^{3/2}R^2}{(\Gamma(1/4))^2\sqrt{\phi^2+{y_0}^2}}\Bigg[{}_3F_2\left(\tfrac{1}{4},\tfrac{3}{4},\tfrac{3}{4};\tfrac{1}{2},\tfrac{5}{4};\frac{\phi^4}{({y_0}^2+\phi^2)^2}\right) \\ & + \frac{\phi^2(\Gamma(1/4))^4}{48\pi^2(\phi^2+{y_0}^2)}\,{}_3F_2\left(\tfrac{3}{4},\tfrac{5}{4},\tfrac{5}{4};\tfrac{3}{2},\tfrac{7}{4};\frac{\phi^4}{({y_0}^2+\phi^2)^2}\right)\Bigg]. \end{aligned} \tag{3.5}$$

The quark energy in (3.2) is found using (2.3) and (3.3). Hence, we find that

$$E \;=\; \frac{1}{\pi}\int_{y_0}^{\infty} \frac{y^2+\phi^2}{\sqrt{(y^2+\phi^2)^2 - ({y_0}^2+\phi^2)^2}}dy\,. \tag{3.6}$$

This integral is divergent because the energy includes the quark bare masses, which diverges as the position of the brane is taken to infinity. We can find the finite potential piece by cutting off the integral at y_{max} and subtracting off y_{max}/π, the sum of the quark masses. Alternatively, we can regularize the integral in (3.6) by replacing the numerator of the integrand by $(\phi^2+y^2)^s$ and analytically continuing to $s=1$. In either case, the result is the same and the quark potential energy is

$$\begin{aligned} E = & -\frac{\sqrt{2\pi}\sqrt{\phi^2+{y_0}^2}}{(\Gamma(1/4))^2}\Bigg[{}_2F_1\left(-\tfrac{1}{4},\tfrac{3}{4};\tfrac{1}{2};\frac{\phi^4}{({y_0}^2+\phi^2)^2}\right) \\ & - \frac{(\Gamma(1/4))^4\phi^2}{16\pi^2(\phi^2+{y_0}^2)}\,{}_2F_1\left(\tfrac{1}{4},\tfrac{5}{4};\tfrac{3}{2};\frac{\phi^4}{({y_0}^2+\phi^2)^2}\right)\Bigg]. \end{aligned} \tag{3.7}$$

The asymptotic behavior of L for large and small y_0 is

$$\begin{aligned} L &= \frac{(2\pi)^{3/2}R^2}{(\Gamma(1/4))^2}\,{y_0}^{-1} \;+\; \mathrm{O}(1/y_0^3) \qquad\qquad y_0 >> \phi \\ L &= -\frac{\sqrt{2}R^2}{\phi}\;\log(y_0) \;+\; \mathrm{O}(1) \qquad\qquad y_0 << \phi\,, \end{aligned} \tag{3.8}$$

while the asymptotic behavior for the potential is

$$\begin{aligned} E &= -\frac{\sqrt{2\pi}}{(\Gamma(1/4))^2}\,y_0 \;+\; \mathrm{O}(1/y_0) \qquad\qquad y_0 >> \phi \\ E &= -\frac{\phi}{\sqrt{2\pi}}\;\log(y_0) \;+\; \mathrm{O}(1) \qquad\qquad y_0 << \phi\,. \end{aligned} \tag{3.9}$$

Hence we see that for small L the quark potential has the Coulombic behavior

$$E \;\approx\; -\;\frac{4\pi^2\sqrt{4\pi gN}}{(\Gamma(1/4))^4}L^{-1}\,, \tag{3.10}$$

while for large L the quark potential has the confining behavior

$$E \approx \frac{1}{2\pi\sqrt{4\pi g N}}\phi^2 L = \frac{\pi}{2}\frac{{m_W}^2}{R^2}L\,. \tag{3.11}$$

The Coulombic behavior is the same as that found for the unbroken phase [2, 3]. This is not surprising; at short distances the energy scale is much larger than the W mass, so one expects the potential to approach the Coulomb potential for the unbroken case.

However, for large L one would still expect to find a Coulombic potential. This is because when the energy scale drops well below that of the W mass, one would expect the W-particles to become irrelevant, and so one should find the Coulombic behavior for an $SU(M)$ ($M = N/2$) super Yang-Mills theory. Hence there should be other geodesics that describe this behavior.

3.2. Geodesics around the symmetry axis

In order to find such geodesics, we start by looking for geodesics that are very close to the straight trajectory. Indeed, to lowest order in perturbation theory we can assume that the motion in the y direction is the same as for the straight trajectory, and take the motion in the x direction to be infinitesimally small.

The equation of motion in the x-direction is then is approximately

$$\frac{d^2x}{d\lambda^2} = -\left.\frac{\partial^2 V(x,y)}{\partial x^2}\right|_{x=0} x = -\frac{2R^4(5\phi^2-(y(\lambda))^2)}{(\phi^2+(y(\lambda))^2)^4}\,x\,, \tag{3.12}$$

where $y(\lambda)$ corresponds to the motion of a straight trajectory. Hence if $y(\lambda)^2 > 5\phi^2$ then the motion in the x direction will accelerate exponentially away from the symmetry axis, $x = 0$. However, if $y(\lambda)^2 < 5\phi^2$ then the there is a restoring force, and the motion is oscillatory. Consider the geodesics that start at $x = 0$ and $y = y_{max} \to \infty$ and return to the same point. Such geodesics cannot exist unless they go close enough to the origin for the x-motion to experience a restoring force (*i.e.* have $y(\lambda)^2 < 5\phi^2$). It follows that if $L << R^2/\phi$ then the only geodesic that starts and finishes on the symmetry axis is the one that runs down the symmetry axis and back. There is thus a phase of this theory in which the $\mathbb{Z}_2$ symmetry is unbroken, and the quark-antiquark pair feel only the full $SU(N)$ gauge theory.

Consider a geodesic whose initial velocity is such that x is still small when it enters the region with $y(\lambda)^2 < 5\phi^2$, and begins to oscillate. For the geodesic to begin and end on the symmetry axis, $x = 0$, the number of half-period oscillations has to be an integer. Hence, these modes will be quantized. We first examine the situation where there are a large number of oscillations, which means that many of the oscillations are very close to the saddle. We want to find the difference in energy between the n and $n+1$ half-periods. The motions are almost identical, except for an extra half-oscillation near the saddle. For this part of the motion y can be approximated as a constant $y \approx 0$, and so using (3.12) we find that the period for this extra half oscillation is

$$\Delta\lambda = \frac{\pi}{\sqrt{10}R^2}\,\phi^3. \tag{3.13}$$

Hence, using (2.7), (2.18) and (3.11), the increase in quark distance between the appearance of n and $n+1$ oscillations is

$$\Delta L = \Delta\lambda R^4\phi^{-4} = \frac{\pi R^2}{\sqrt{10}}\,\phi^{-1}. \tag{3.14}$$

and the change in the potential energy is

$$\Delta E = \frac{\pi}{2\sqrt{10}}\,\frac{\phi}{\pi}. \tag{3.15}$$

Thus, a trajectory with an extra half-oscillation appears when the energy is increased by an amount that is slightly less than half the W mass, $m_W = \phi/\pi$.

Once we have isolated a trajectory with a *fixed number*, n, of small half-oscillations, we can increase L and find the asymptotic behavior for this trajectory. As L increases, the trajectory will move further and further up the two peaks, or closer to the singularities of V, until it reaches the limit where the trajectory is the concatenation of a straight line coming in from $(x,y) = (0,-\infty)$ to the peak at $(\phi,0)$ and then $n-1$ lines going back and forth through the saddle between the peaks and finally a straight line from one of the peaks back to asymptotic infinity. Figures 1 and 2 show the L behavior for geodesics with 2 and 6 half-oscillations. Hence the total energy of this configuration is the sum of the quark masses plus $n-1$ times the W mass ϕ/π. In other words, as $L\to\infty$, we find that we are left with the two heavy quarks plus $n-1$ W-particles.

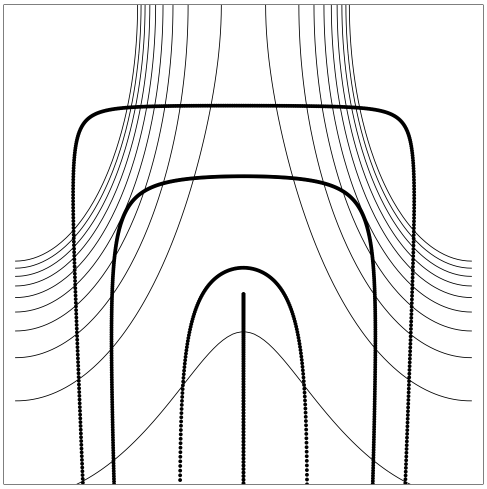

Figure 1: Geodesics with two half oscillations overlayed on a contour map of the potential. As L is increased, the geodesic approaches the concatenation of the BPS geodesics of two quarks and a W. The potential peaks are at the upper corners. The straightline trajectory in the figure is the small oscillation limit for two half-oscillations.

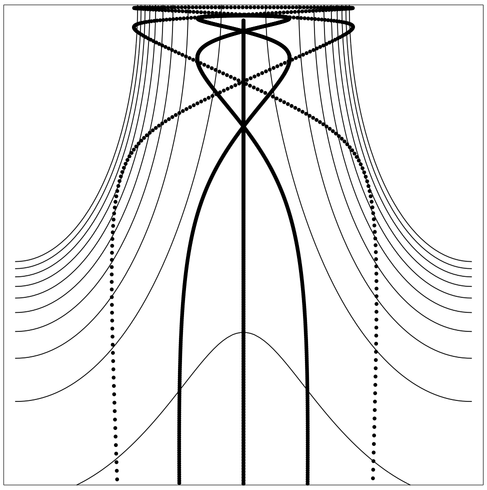

Figure 2: Geodesics with six half oscillations. As L is increased, the geodesic approaches the concatenation of the BPS geodesics of two quarks and a five Ws.

It is also instructive to reverse this argument, and start with the heavy quarks separated by a large distance but with n W's in between. As L becomes finite, the trajectory moves off the peaks and has $n+1$ smooth half-oscillations. Hence this corresponds to a situation with heavy quarks and n intermediate W-particles. As L is decreased the oscillations become smaller and smaller, until we hit a critical distance L_n, where the oscillations disappear all together and the trajectory becomes our original straight-line trajectory. As we decrease L further, the straightline trajectory becomes shorter until it coincides with the trajectory with n oscillations at $L = L_{n-1}$. Obviously, this trajectory corresponds to the case with heavy quarks and $n-1$ W-particles. Therefore, when $L = L_{n-1}$, the binding energy for the last W among n W-particles is the W mass ϕ/π.

We can make a reasonable estimate of the strength of the Coulombic potential on the branch with n W-particles. If L is large, we can see from the figures that the oscillations get very close to the peaks. Moreover, each oscillation reaches roughly the same distance ϵ away from the peak with its velocity becoming quite small. Hence we can use the results of [2, 3] to estimate L and E. For each half-oscillation there is an ascent and a descent of a peak. Hence, L as a function of ϵ is approximately

$$L \;=\; \frac{2(\pi)^{3/2}(n+1)R^2}{(\Gamma(1/4))^2}\,\epsilon^{-1} \;+\; \mathrm{O}(1/\epsilon^2)\,, \tag{3.16}$$

where there is a factor of $n+1$ coming from the $n+1$ half-oscillations and a factor of $1/\sqrt{2}$, since the effective gauge group at each peak is $SU(N/2)$. Likewise, the potential energy is

$$E \;=\; -\,\frac{\sqrt{2\pi}(n+1)}{(\Gamma(1/4))^2}\,\epsilon \;+\; \mathrm{O}(1) \tag{3.17}$$

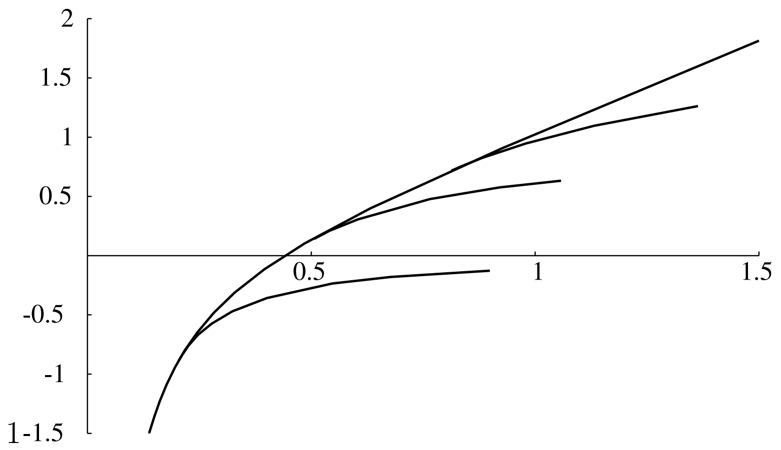

Figure 3: Plot of E vs. L for the saddle branch and the first three Coulomb branches. E is plotted in units of the W mass, m_W, and L is in units of R^2/m_W. The first and third branches are doubly degenerate. The second branch is a $\mathbb{Z}_2$ symmetric branch.

and hence E as a function of L is

$$E \;\approx\; -\;\frac{4\pi^2(n+1)^2\sqrt{2\pi gN}}{(\Gamma(1/4))^4}\;L^{-1}\,. \tag{3.18}$$

This Coulombic behavior can be understood as follows: we can assume that there are n intermediate W-particles equally spaced between the quarks. In the large N limit each W feels a Coulombic attraction only with its two neighbors (the W-particles on the end feel an attraction to the neighboring quark), where the strength of this interaction is the Coulombic quark anti-quark interaction for gauge group $SU(N/2)$ but for a separation of $L/(n+1)$. Since there are $n+1$ interacting pairs, we expect the total Coulomb potential in (3.18). Since the W-particles can move, we should expect that these branches are unstable because of a Coulomb instability. We will discuss this further in the next subsection.

Figure 3 shows the branch of straight geodesics, which becomes linear for large L, and the first three Coulomb branches. The derivatives dE/dL are continuous where the Coulomb branches attach to the saddle branch, but d^2E/dL^2 is not. The lowest Coulomb branch is the quark-antiquark branch with no intermediate W-particles. This branch is doubly degenerate, since a trajectory on this branch goes up one of the two peaks. The second Coulomb branch is a quark and antiquark with an intermediate W-particle. There is only one branch since these trajectories are $\mathbb{Z}_2$ symmetric. Finally the third Coulomb branch, which is also doubly degenerate, has two W-particles between the quark-antiquark pair.

It is instructive to compare the linear behavior of the straight trajectories to the Coulombic behavior of (3.18) when $L = L_n$ and n is large. In the linear regime (3.11) yields

$$\frac{dE}{dL} \;=\; \frac{\pi {m_W}^2}{2R^2} \;\approx\; 1.571\frac{{m_W}^2}{R^2}\,, \tag{3.19}$$

where $m_W = \phi/\pi$. For large n (3.14) implies $L_n \approx \frac{\pi R^2 n}{\sqrt{10}}\phi^{-1}$, and hence the energy on the $n^{\rm th}$ Coulomb branch, (3.18), yields

$$\frac{dE}{dL} \approx \frac{40\pi^2}{\sqrt{2}(\Gamma(1/4))^4}\frac{{m_W}^2}{R^2} \approx 1.616\frac{{m_W}^2}{R^2}\,. \qquad (3.20)$$

Hence, we see that the Coulomb behavior of this branch persists all the way down to where it merges with the linear branch.

We have thus seen that infinitesimal perturbations of the symmetry axis geodesic yield geodesics corresponding to a quark, anti-quark and n intermediate W-particles experiencing a net attractive Coulombic potential. We have also seen that there is a non-trivial threshold to go from n to $n+1$ intermediate W-particles. This means that we should view the symmetry axis geodesic as representing the process of adiabatically separating the quark and anti-quark. As the separation gets larger, the number of W-particles that can be produced between the quarks increases. It is because of this increase in the W number that the energy does not undergo the Coulombic fall-off, but instead exhibits the confining behavior (3.11).

A small perturbation away from the straight geodesic leads to a selection of a definite W-number, and then further adiabatic separation of the quark and anti-quark preserves this number, but brings the state closer and closer to the BPS threshhold. It is easy to see that once a definite particle number has been selected, the confining behavior of E must at least turn over, and revert to a Coulombic decay. The reason is that if the energy increased without bound, then *because the particle number is fixed*, the energy will ultimately become far larger than the total mass of all the W-particles between the quarks, and once again scale invariance will effectively be restored. We thus have the following picture of $E(L)$ for all these states: the straight geodesic gives rise to and energy function $E_*(L)$ that is akin to the Cornell potential, starting as $E_*(L) \sim -1/L$, and then crossing over to $E_*(L) \sim L$. At finite, and ultimately regular intervals, the graph of $E(L)$ versus L bifurcates: the confining potential, $E_*(L)$ continues, but a new branch, $E_n(L)$, drops away and corresponds to a quark, anti-quark and n W-particles. This branch then follows the Coulombic behavior (3.18), shifted by the mass, $n\phi/\pi$, of n W-particles.

3.3. QCD configurations and stability

As we discussed in subsection 3.2, the physical picture for the $(n+1)^{\rm th}$ Coulomb branch is that of n W-particles equally spaced between the quarks. This picture is confirmed by considering the energy density, ρ, along the string, which is given by (2.8). For a particle oscillating between the peaks, the QCD energy density gets very small as the geodesics approach the peaks, and becomes finite, or large as the geodesic passes across the symmetry axis. Indeed, as the particle moves higher and higher up the peaks it spends most of its time on the peaks, and very little time in between. Thus the QCD energy density becomes very tightly peaked, and if the particle executes n half-periods then there are $n+1$ such peaks, the first and last being the quark and anti-quark, and the remaining $n-1$ being W-particles. For large n, the particle motion approaches the saddle and the oscillations become regularly spaced and so the W-particles become very regularly strung out along the QCD string between the quark and anti-quark. Figure 3.3 shows plots of the string energy density for various values of L on the Coulomb branch with six intermediate W-particles. The plots show that the W-particles quickly localize as the quarks are pulled apart.

density6.eps,height=1.2in Plots of ρ vs. ℓ for the Coulomb branch with six intermediate W's. ρ is in units of ${m_W}^2/R^2$ and ℓ is in units of R^2/m_W. The quark separations are (a) $L = 2.19R^2/m_W$ (b) $L = 2.72R^2/m_W$ and (c) $L = 3.37R^2/m_W$. The critical value for the appearance of this branch is $L \approx 2.07R^2/m_W$.

Thus these oscillating geodesics correspond to W-particles strung out like beads. For the geodesic that runs up to the saddle without oscillating, the W-particles are smeared out along the length of the string, and as the oscillations increase the W-particles localize to a greater and greater degree. Because of this attractive Coulomb potential between the beads on the string, we see that this configuration is very unstable, and so we may infer that the linear flux tube is also unstable.

We can also see this instability of the flux tube using perturbative analysis: Consider geodesics that run straight up the saddle and look for small perturbations δx in the x direction. Using equations (2.3) and (2.6) we see that perturbation of the energy is

$$\begin{aligned} \delta E &= \frac{1}{4\pi}\int \frac{dz}{\sqrt{\left(\frac{\partial y}{\partial z}\right)^2 + H^{-1}}}\left(\left(\frac{\partial \delta x}{\partial z}\right)^2 - \frac{1}{2}H^{-2}\frac{\partial^2 H}{\partial x^2}\delta x^2\right) \\ &= \frac{1}{4\pi}\int d\lambda \left(\left(\frac{\partial \delta x}{\partial \lambda}\right)^2 - \frac{\partial^2 V}{\partial x^2}\delta x^2\right) . \end{aligned} \tag{3.21}$$

The variation of the length is

$$\delta L \;=\; 2\int d\lambda \left(\frac{\partial^2 V}{\partial x^2}\delta x^2 + \frac{\partial V}{\partial y}\delta y\right) \tag{3.22}$$

and so a quadratic variation in x can be compensated for by a linear variation in y in order that the length remain fixed. Thus, we see that the string configuration is unstable if the operator

$$\mathcal{L} \;=\; -\frac{\partial^2}{\partial \lambda^2} - \frac{\partial^2 V}{\partial x^2} \tag{3.23}$$

has negative eigenvalue solutions. But we know from (3.12) that a Coulomb branch occurs precisely when this operator has a zero eigenvalue. Therefore, every time the linear branch passes by a Coulomb branch, an eigenvalue of $\mathcal{L}$ changes sign. Hence if $L_n < L < L_{n+1}$ then the linear branch has n unstable modes. In terms of W-particles, we expect an unstable mode for each particle and so the picture fits.

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To summarize, we see that the linear branch is unstable because the flux tube is energetic enough to pop out intermediate W-particles between the quarks. The appearance of the W-particles is marked by a localization of the energy density along the string between the quarks. The presence of these W-particles then destabilizes the configuration because of a Coulomb instability. We have seen in (3.19) and (3.20) that the Coulomb behavior of the n branch persists all the way down to $L = L_n$, so we expect very little softening of this instability on the linear branch.

When $L < L_1$, then the branch of straight trajectories is stable and is in fact the only solution that extremizes the energy. In this case the $\mathbb{Z}_2$ symmetry is unbroken and the quark and anti-quark are in the fundamental of $SU(N)$. When $L > L_1$ then the stable branch is the first Coulomb branch which is doubly degenerate. The degeneracy of this branch reflects the fact that the quark and anti-quark are fundamentals of one of the $SU(N/2)$ groups. Thus, if we vary L between values that are less than L_1 and greater than L_1 such that we always remain on the stable branch, we see that there is a third order transition in that d^2E/dL^2 is not continuous at $L = L_1$. At this point the $\mathbb{Z}_2$ symmetry is broken.

4. Asymmetric and more general situations

4.1. Trajectories off the symmetry axis

Up to this point we have been assuming that the $D3$-brane at infinity is on the symmetry axis defined by the two sets of $D3$-branes. For this configuration the heavy quarks are in the fundamental representation for one of the two $SU(N/2)$ groups and both types of quarks have the same BPS mass. If the $D3$-brane at infinity is not on this axis, then the quarks will have unequal masses. If we denote the lighter quark by q and the heavier quark by q', then their mass difference is $\Delta m = \phi\cos(\theta)/\pi$, where θ is the angle away from the $D3$ brane axis. So if $\theta = 0$ degrees then the difference in quark mass is the mass of the W.

We now consider trajectories that are at an angle θ at asymptotic infinity. Qualitatively, we expect to find behavior similar to the symmetric case for large L. That is we expect to find trajectories corresponding to the Coulomb potentials for $q\overline{q}$ with some number n of intermediate W-particles. As L is decreased, the energy of this configuration will match the potential energy for $q'\overline{q}$ with $n-1$ W-particles. Then as L is decreased even further this will match the potential energy for $q'\overline{q}'$ with $n-2$ W-particles. However, to make the next step to $n-3$ W-particles, we have to first jump down to $q\overline{q}$ with $n-2$ W-particles, which releases a finite jump in energy because of the mass difference between the q and q'. Hence these last two geodesics will not be continuously connected.

Hence, we expect to find a lone geodesic that corresponds to $q\overline{q}$ and which exists for any value of L. All other geodesics will come in bifurcating triple families. That is, the n^{th} "triple" will correspond to $q\overline{q}$ with $n+2$ W-particles (n is even), $q'\overline{q}$ with

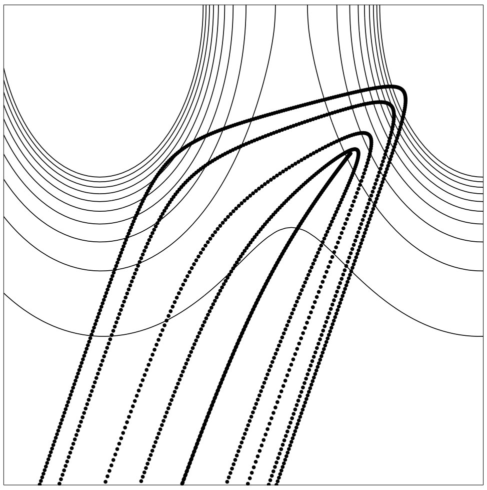

Figure 4: Geodesics with an incoming angle of 120 degrees that loop and cross the symmetry axis twice. For large L, the trajectory corresponds to a $q'\overline{q}$ pair and an intermediate W. There is a minimum value of L where the loop contracts to a line that retraces itself. At this value, the trajectory becomes identical with the trajectory for $q\overline{q}$ pair and two W-particles.

$n+1$ W-particles and $q'\overline{q}'$ and n W-particles. These three families of geodesics will be distinct for L greater than some critical amount. At very large L, the strength of the Coulombic force will be different for the different branches, and the energy curves will have an asymptotic separation of the W-boson mass. As L decreases the energy curves, $E(L)$, must merge, as outlined above, and $E(L)$ must be singular (have a cusp) at the last merger.

One can easily see how this picture of triples emerges from the symmetric picture considered in section 3. Consider the functions, $E_*(L)$, and $E_n(L)$ described at the end of the last section. For n even the relevant geodesics ultimately run up one peak or the other, come to a halt and reverse their course, whereas for n odd, the geodesics never come to a halt and are symmetric between the peaks as in figures 1 and 2. The energy functions $E_n(L)$, for n even, are thus doubly degenerate, corresponding to $q\overline{q}$ and n W's, or $q'\overline{q}'$ and n W's. This degeneracy is lifted when we split the degeneracy of the quark masses, and the bifurcating curves $E_*(L)$ and $E_n(L)$ split into triples along the $E_n(L)$ curves for n even.

Returning to the asymmetric situation, we can locate the critical values for the length at which mergers occur by examining the general behavior of the geodesics as a critical point is approached. As we saw in the previous section, critical points may appear if a geodesic becomes unstable due to a small oscillation. A singularity of this type is shown in figure 4. In this figure we will denote the quark associated with the left-hand peak by q and the other quark by q'. We are considering geodesics with an incoming angle of 120 degrees, and there is one in particular that rolls up the potential, stops, and then rolls back down. This geodesic is the limit of geodesics that roll up one peak, move to the other peak and then roll back down. The latter class of geodesic may

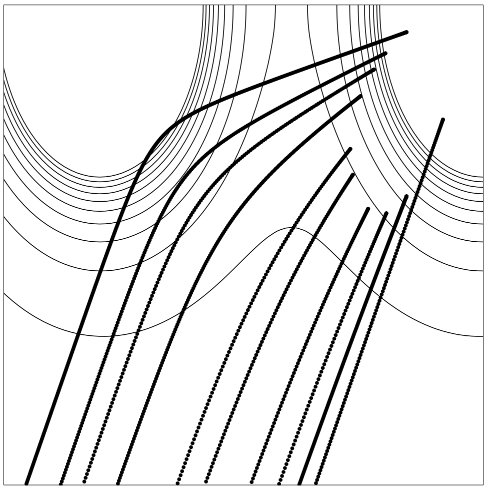

Figure 5: Geodesics with an incoming angle of 120 degrees that cross the symmetry axis twice. The trajectories stop at some point and roll back in the opposite direction. The figure shows two branches for a given L. For large L, one branch corresponds to $q\overline{q}WW$ while the other branch corresponds to a $q'\overline{q}'$ pair. The fifth trajectory from the left is the limit of the loop trajectories in figure 4. The third trajectory from the right corresponds to the trajectory with minimum length.

be thought of as the state $q'\overline{q}W$. The limit where this loop closes is a singular point. For this critical value of L, the $q'\overline{q}W$ trajectory merges into the $q\overline{q}WW$ trajectory. This second family of geodesics may be seen in upper part of figure 5: these geodesics roll up the left peak, roll over to the right, and then reverse their course. In terms of QCD, the larger Coulomb potential for $q\overline{q}WW$ compensates for the larger BPS mass of its constituents at the critical point. For L larger than this critical value one can be on either the $q\overline{q}WW$ or the $q'\overline{q}W$ family, while for L less than the critical value these two families are merged and are indistinguishable.

There is another type of singularity which is not really a singularity of the geodesics, but results in a singularity for the large N Super Yang-Mills. Suppose we have a family of geodesics parameterized by some quantity σ. Then E and L are both functions of σ. However, E and L can be multivalued and if so there is some point σ_0 where $\partial_\sigma E = 0$ and thus, by arguments given in section 2, $\partial_\sigma L = 0$. If we plot E as a function of L for these families of geodesics, we would find a cusp where the σ derivatives vanish. Hence this is a singularity and in fact corresponds to the minimum L and E for these families of geodesics. Figure 5 shows a family of geodesics that have an incoming angle of 120 degrees and roll up the second peak, stop and then roll back down. As we discussed this family contains a transition from $q\overline{q}WW$ to a merger with $q'\overline{q}W$. Continuing beyond this we can smoothly deform these geodesics so that the "rest point" of the particle moves down the peak and then moves back up it, finally representing a $q'\overline{q}'$ state. Clearly this family of geodesics will have a minimum value of L since L diverges as the

Figure 6: Plot of E vs. L for the $q\overline{q}$ branch, and the two lowest sets of triples. As L is decreased the upper two branches in a triple merge merge. At even smaller L this branch merges with the bottom branch in the triple. To decrease L even further requires a finite jump in energy to a lower triple.

rest point of the particle moves up the peak. At the minimum of L, the $q'\overline{q}'$ trajectory merges with the $q'\overline{q}W$ trajectory. Note that this critical point does not necessarily occur when the particle energy is minimized.

We can also see in figure 5 that there is a trajectory in this class of geodesics where the particle energy μ is minimized. Near this point, $\frac{dE}{dL} = 1/\sqrt{2\mu}$ is roughly constant, so the potential is linear in L. Hence there is confining like behavior for trajectories in the neighborhood of this point. These trajectories are the merged $q'\overline{q}W$ trajectories. This appears to be a general feature. That is, when a critical value L_n is reached such that the $q\overline{q}W^{2n}$ trajectories merge with the $q'\overline{q}W^{2n-1}$ trajectories, then for $L < L_n$ the potential is roughly linear in L. Figure 6 shows the lowest Coulomb branch and the first two triples. In the lowest triple, $q\overline{q}WW$ merges with $q'\overline{q}W$ which then merges with $q'\overline{q}'$. Comparing Figure 6 with figure 3, one sees that the effect of the unequal mass is to pull the degenerate branches in the symmetric case apart, leaving isolated sets of branches.

Thus once one moves off the symmetry axis, the interesting features of the more symmetric situation survive, but in a more limited form. For example, there still appears to be a nearly linear potential for a range of separations, and in this range the number of W's is changing, but the asymmetry limits the amount of "continuous" change in the particle number, and hence the behavior ultimately bifurcates into various Coulombic limits.

The lone $q\overline{q}$ branch is obviously stable for any value of L since it is the lowest energy state. At L less than some critical value, L_c, this is the only available branch. For $L > L_c$ a bifurcated triple branch appears. The lowest leg of the triple is the $q'\overline{q}'$ branch. While this branch has more energy than the $q\overline{q}$ branch because of the higher quark mass, this configuration is still a local minimum. To see this, note that in order for a q' to switch to a q, we need to pop a W near the quark, but the W will feel a net attractive force back to the quark. If we could pull the W far enough away from the quark, it would eventually feel an attractive force toward the $\overline{q}'$ antiquark, and then these two would combine to form a $\overline{q}$ antiquark. Hence, we expect the $q'\overline{q}'$ branch to be stable, but the branch destabilizes when it merges with the $q'\overline{q}W$ branch, which occurs at the cusp at $L = L_c$. There is then a finite change in energy as the configuration jumps down to the low branch. All other branches are unstable.

It is now relatively easy to see the general structure of what will happen in breaking $SU(N)$ to $SU(N_1) \times SU(N_2)$. First, there is still a saddle point, and thus there will be a "confining" geodesic running up to it. The confining geodesic will approach the two peaks of the potential V at some angle other than 90 degrees. One can view this angle as the asymmetry in the potential being compensated for by an asymmetry in the quark masses. There will be small oscillations about the confining geodesic, and much of our earlier discussion will go through. Indeed, it is easy to convince onself that the asymmetric potential, V, will result in families of geodesics that are deformations of the ones considered earlier. The corresponding conclusions about the quark-antiquark potential will thus be completely analogous.

If one breaks $SU(N)$ to p simple factors one can have a much richer saddle-point structure in the potential V. This will lead to different "confining geodesics," and to far more complex structure. In the $\mathbb{Z}_p$ symmetric breaking to $(SU(N/p))^p$, where the $D3$-branes sit on the vertices of a regular p-gon, one might expect that the Z_p symmetric "confining geodesic" running along the symmetry axis of the p-gon would become more stable if the curvature in the saddle is softened.

5. Final comments

$N = 4$ supersymmetric gauge theories are conformal and cannot confine, and yet we see geodesics that have some confining behavior. There is no real conflict here since the "confining geodesics" are classically and quantum mechanically very unstable, and the corresponding QCD state will destabilize by popping out W-particles between the quarks, which are in turn swallowed by the quarks because of the Coulomb attraction. In the end we are left with two quarks feeling a Coulombic force. Of course linear flux tubes are not stable for real QCD either. If two quarks are pulled far enough apart, the flux tube destabilizes by popping a shower of hadrons out of the vacuum.

On the more technical level, we have also found a simple relation for the interquark force in terms of a classical particle's energy. Through this we have shown that the geodesic prescription of [2, 3] always leads to an attractive force. It also gives a method for finding transitions in particle number, and for finding confining families of geodesics.

Acknowledgments

This work was supported in part by funds provided by the DOE under grant number DE-FG03-84ER-40168.

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